Vertex Coloring with Communication and Local Memory Constraints in Synchronous Broadcast Networks

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Abstract

The vertex coloring problem has received a lot of attention in the context of synchronous round-based systems where, at each round, a process can send a message to all its neighbors, and receive a message from each of them. Hence, this communication model is particularly suited to point-to-point communication channels. Several vertex coloring algorithms suited to these systems have been proposed. They differ mainly in the number of rounds they require and the number of colors they use.

This paper considers a broadcast/receive communication model in which message collisions and message conflicts can occur (a collision occurs when, during the same round, messages are sent to the same process by too many neighbors; a conflict occurs when a process and one of its neighbors broadcast during the same round). This communication model is suited to systems where processes share communication bandwidths. More precisely, the paper considers the case where, during a round, a process may either broadcast a message to its neighbors or receive a message from at most m of them. This captures communication-related constraints or a local memory constraint stating that, whatever the number of neighbors of a process, its local memory allows it to receive and store at most m messages during each round. The paper defines first the corresponding generic vertex multi-coloring problem (a vertex can have several colors). It focuses then on tree networks, for which it presents a lower bound on the number of colors K that are necessary (namely, $K = \lceil \frac{\Delta}{m} \rceil + 1$, where Δ is the maximal degree of the communication graph), and an associated coloring algorithm, which is optimal with respect to K.

Keywords: Broadcast/receive communication, Bounded local memory, Collision-freedom, Conflict-freedom, Distributed algorithm, Message-passing, Multi-coloring, Network traversal, Scalability, Synchronous system, Tree network, Vertex coloring.

1 Introduction

Distributed message-passing synchronous systems From a structural point of view, a message-passing system can be represented by a graph, whose vertices are the processes, and whose edges are the communication channels. It is naturally assumed that the graph is connected.

Differently from asynchronous systems, where there is no notion of global time accessible to the processes, synchronous message-passing systems are characterized by upper bounds on message transfer delays and processing times. Algorithms for such systems are usually designed according to the round-based programming paradigm. The processes execute a sequence of synchronous rounds, such that, at every round, each process first sends a message to its neighbors, then receives messages from them, and finally executes a local computation, which depends on its local state and the messages it has received. The fundamental synchrony property of this model is that every message is received in the round in which it was sent. The progress from one round to the next is a built-in mechanism provided by the model. Algorithms suited to reliable synchronous systems can be found in several textbooks (e.g., [19, 21])[1]. When considering reliable synchronous systems, an important issue is the notion of local algorithm. Those are the algorithms whose time complexity (measured by the number of rounds) is smaller than the graph diameter [1, 17].

Distributed graph coloring in point-to-point synchronous systems One of the most studied graph problems in the context of an n-process reliable synchronous system is the vertex coloring problem, namely any process must obtain a color, such that neighbor processes must have different colors (distance-1 coloring), and the total number of colors is reasonably "small". More generally, the distance-k coloring problem requires that no two processes at distance less or equal to k, have the same color. When considering sequential computing, the optimal distance-1 coloring problem is NP-complete [12].

When considering the distance-1 coloring problem in an n-process reliable synchronous system, it has been shown that, if the communication graph can be logically oriented such that each process has only one predecessor (e.g., a tree or a ring), $O(\log^* n)$ rounds are necessary and sufficient to color the processes with at most three colors [10, 17][2]. Other distance-1 coloring algorithms are described in several articles (e.g. [3, 5, 14, 16]). They differ in the number of rounds they need and in the number of colors they use to implement distance-1 coloring. Let Δ be the maximal degree of the graph (the degree of a vertex is the number of its neighbors). Both algorithms in [3, 5] color the vertices with $(\Delta+1)$ colors. The first one requires $O(\Delta + \log^* n)$ rounds, while the second one uses $O(\log \Delta)$ rounds. An algorithm is described in [14] for trees, which uses three colors and $O(\log^* n)$ rounds. Another algorithm presented in the same paper addresses constant-degree graphs, and uses $(\Delta + 1)$ colors and $O(\log^* n)$ rounds. The algorithm presented in [16] requires $O(\Delta \log \Delta + \log^* n)$ rounds. These algorithms assume that the processes have distinct identities[3], which define their initial colors. They proceed iteratively, each round reducing the total number of colors. Distributed distance-2 and distance-3 coloring algorithms, suited to various synchronous models, are presented in [6, 8, 9, 11, 13, 15].

Motivation and content of the paper The previous reliable synchronous system model assumes that there is a dedicated (e.g., wired) bi-directional communication channel between each pair of neighbor processes. By contrast, this paper considers a broadcast/receive communication model in which there is no dedicated communication medium between each pair of neighbor processes. This covers practical system deployments, such as wireless networks and sensor networks. In such networks, the prevention

[1] The case where processes may exhibit faulty behaviors (such as crashes or Byzantine failures) is addressed in several books (e.g., [2, 18, 19, 20]).

[2] $\log^* n$ is the number of times the function $\log$ needs to be iteratively applied in $\log(\log(\log(...(\log n))))$ to obtain a value ≤ 2. As an example, if n is the number of atoms in the universe, $\log^* n \backsimeq 5$.

[3] Some initial asymmetry is necessary to solve *breaking symmetry problems* with a deterministic algorithm.

of collisions (several neighbors of the same process broadcast during the same round), or conflicts (a process and one of its neighbors issue a broadcast during the same round), does not come for free. In particular, round-based algorithms that seek to provide deterministic communication guarantees in these systems must be collision and conflict-free (*C2*-free in short).

We are interested in this paper to offer a programming model in which, at any round, a process can either broadcast a message to its neighbors (conflict-freedom), or receive messages from at most m of its neighbors (m-collision-freedom). This means that we want to give users a round-based programming abstraction guaranteeing conflict-freedom and a weakened form of collision-freedom, that we encapsulate under the name C2m-freedom (if $m = 1$, we have basic C2-freedom).

The ability to simultaneously receive messages from multiple neighbors can be realized in practice by exploiting multiple frequency channels[4]. The parameter $m \geq 1$ is motivated by the following observations. While a process (e.g., a sensor) may have many neighbors, it can have constraints on the number of its reception channels, or constraints on its local memory, that, at each round, allow it to receive and store messages from only a bounded subset of its neighbors, namely m of them ($m = 1$, gives the classic C2-free model, while $m \geq \Delta$ assumes no collision can occur as in the classic broadcast/receive model presented previously). This "bounded memory" system parameter can be seen as a scalability parameter, which allows the degree of a process (number of its neighbors) to be decoupled from its local memory size.

C2m-freedom can be easily translated as a coloring problem, where any two neighbors must have different colors (conflict-freedom), and any process has at most m neighbors with the same color (m-collision-freedom). Once such a coloring is realized, message consistency is ensured by restricting the processes to broadcast messages only at the rounds associated with their color. While it is correct, such a solution can be improved, to allow more communication to occur during each round. More precisely, while guaranteeing C2m-freedom, it is possible to allow processes to broadcast at additional rounds, by allocating multiple colors to processes. From a graph coloring point of view, this means that, instead of only one color, a set of colors can be associated with each process, while obeying the following two constraints: (a) for any two neighbor processes, the intersection of their color sets must remain empty; and (b) given any process, no color must appear in the color sets of more than m of its neighbors.

We call *Coloring with Communication/Memory Constraints* (CCMC) the coloring problem described above. More precisely, this problem is denoted CCMC$(n, m, K, \geq 1)$, where n is the number of processes (vertices), m is the bound on each local memory (bound on the number of simultaneous communication from a reception point of view), and K the maximal number of colors that are allowed. "≥ 1" means that there is no constraint on the number of colors that that can be assigned to a process. CCMC$(n, m, K, 1)$ denotes the problem instance where each process is assigned exactly one color. From a technical point of view, the paper focuses on tree networks. It presents a lower bound on the value of K for these communication graphs, and an algorithm, optimal with respect to K, which solves both instances of CCMC.

Roadmap The paper is made up of 7 sections. Section 2 presents the underlying system model. Section 3 formally defines the CCMC problem. Then, considering tree networks, whose roots are dynamically defined, Section 4 presents a lower bound on K for CCMC$(n, m, K, 1)$ and CCMC$(n, m, K, \geq 1)$ to be solved. Section 5 presents then a K-optimal algorithm solving CCMC$(n, m, K, \geq 1)$. (from which a solution to CCMC$(n, m, K, 1)$ can be easily obtained.) Section 6 presents a proof of the algorithm. Finally, Section 7 concludes the paper.

[4]Depending on the underlying hardware (e.g., multi-frequency bandwidth, duplexer, diplexer), variants of this broadcast/receive communication pattern can be envisaged. The algorithms presented in this paper can be modified to take them into account.

2 Synchronous Broadcast/Receive Model

Processes, initial knowledge, and the communication graph The system model consists of n sequential processes denoted p_1, ..., p_n, connected by a connected communication graph. When considering a process p_i, $1 \leq i \leq n$, the integer i is called its index. Indexes are not known by the processes. They are only a notation convenience used to distinguish processes and their local variables.

Each process p_i has an identity id_i, which is known only by itself and its neighbors (processes at distance 1 from it). The constant $neighbors_i$ is a local set, known only by p_i, including the identities of its neighbors (and only them). In order for a process p_i not to confuse its neighbors, it is assumed that no two processes at distance less than or equal to 2 have the same identity. Hence, any two processes at distance greater than 2 can have the very same identity.

Δ_i denotes the degree of process p_i (i.e. $|neighbors_i|$) and Δ denotes the maximal degree of the graph ($\max\{\Delta_1, \cdots, \Delta_n\}$). While each process p_i knows Δ_i, no process knows Δ (a process p_x such that $\Delta_x = \Delta$ does not know that Δ_x is Δ).

Timing model Processing durations are assumed equal to 0. This is justified by the following observations: (a) the duration of local computations is negligible with respect to message transfer delays, and (b) the processing duration of a message may be considered as a part of its transfer delay.

Communication is synchronous in the sense that there is an upper bound D on message transfer delays, and this bound is known by all the processes (global knowledge). From an algorithm design point of view, we consider that there is a global clock, denoted $CLOCK$, which is increased by 1, after each period of D physical time units. Each value of $CLOCK$ defines what is usually called a *time slot* or a *round*.

Communication operations The processes are provided with two operations denoted broadcast() and receive(). A process p_i invokes broadcast TAG(m) to send the message m (whose type is TAG) to its neighbors. It is assumed that a process invokes broadcast() only at a beginning of a time slot (round). When a message TAG(m) arrives at a process p_i, this process is immediately warned of it, which triggers the execution of the operation receive() to obtain and process the message. Hence, a message is always received and processed during the time slot –round– in which it was broadcast.

From a linguistic point of of view, we use the two following **when** notations when writing algorithms, where predicate is a predicate involving $CLOCK$ and possibly local variables of the concerned process.

when TAG(m) **is received do** communication-free processing of the message.
when predicate **do** code entailing at most one broadcast() invocation.

Message collision and message conflict in the m-bounded memory model As announced in the Introduction, there is no dedicated communication medium for each pair of communicating processes, and each process has local communication and memory constraints such that, at every round, it cannot receive messages from more than m of it neighbors. If communication is not controlled, "message clash" problems can occur, messages corrupting each other. Consider a process p_i these problems are the following.

- If more than m neighbors of p_i invoke the operation broadcast() during the same time slot (round), a message *collision* occurs.
- If p_i and one of its neighbors invoke broadcast() during the same time slot (round), a message *conflict* occurs.

As indicated in the introduction, an aim of coloring is to prevent message clashes from occurring, i.e., in our case, ensures C2m-freedom. Let us observe that a coloring algorithm must itself be C2m-free.

3 The Coloring with Communication/Memory Constraints Problem

Definition of the CCMC problem Let $\{p_1, \cdots, p_n\}$ be the n vertices of a connected undirected graph. As already indicated, $neighbors_i$ denotes the set of the neighbors of p_i. Let the color domain be the set of non-negative integers, and m and K be two positive integers. The aim is to associate a set of colors, denoted $colors_i$, with each vertex p_i, such that the following properties are satisfied.

- Conflict-freedom. $\forall i, j : (p_i \text{ and } p_j \text{ are neighbors}) \Rightarrow colors_i \cap colors_j = \emptyset$.
- m-Collision-freedom. $\forall i, \forall c : |\{j : p_j \in neighbors_i \wedge c \in colors_j\}| \leq m$.
- Efficiency. $|\cup_{1 \leq i \leq n} colors_i| \leq K$.

The first property states the fundamental property of vertex coloring, namely, any two neighbors are assigned distinct colors sets. The second property states the m-constraint coloring on the neighbors of every process, while the third property states an upper bound on the total number of colors that can be used.

As indicated in the Introduction, this problem is denoted CCMC$(n, m, K, 1)$ if each color set is constrained to be a singleton, and CCMC$(n, m, K, \geq 1)$ if there is no such restriction.

Example An example of such a multi-coloring of a 21-process network, where $\Delta = 10$, and with the constraint $m = 3$, is given in Figure 1. Notice that $K = \lceil \frac{\Delta}{m} \rceil + 1 = 5$ (the color set is $\{0, 1, 2, 3, 4\}$).

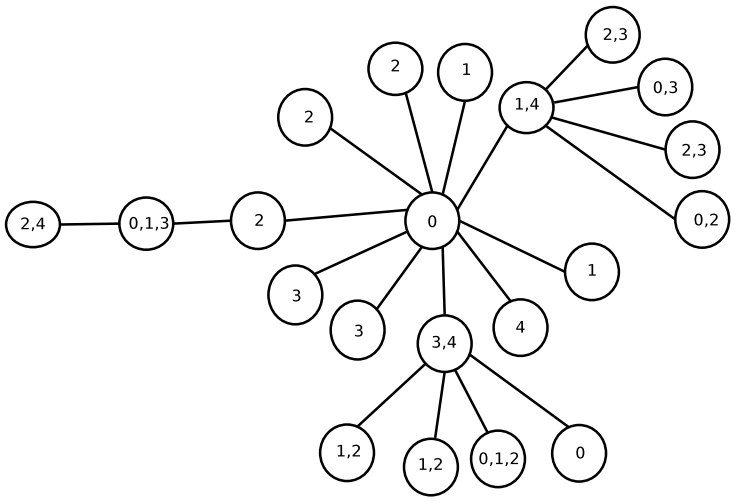

Figure 1: Multi-coloring of a 21-process 10-degree tree with the constraint $m = 3$ (5 colors)

Particular instances The problem instance CCMC$(n, \infty, K, 1)$ is nothing other than the classical vertex coloring problem, where at most K different colors are allowed ($m = \infty$ states that no process imposes a constraint on the colors of its neighbors, except that they must be different from its own color). The problem instance CCMC$(n, 1, K, 1)$ is nothing other than the classical distance-2 coloring problem (vertices at distance ≤ 2 have different colors).

Using the colors The reader can easily see that CCMC$(n, m, K, \geq 1)$ captures the general coloring problem informally stated in the introduction. Once a process p_i has been assigned a set of colors $colors_i$, at the application programming level, it is allowed to broadcast a message to neighbors at the rounds (time slots) corresponding to the values of $CLOCK$ such that $(CLOCK \bmod K) \in colors_i$.

4 CCMC($n, m, K, \geq 1$) in a Tree Network: Lower Bounds

4.1 An impossibility result

Considering tree networks, this section presents a lower bound on K: neither CCMC($n, m, K, 1$), nor CCMC($n, m, K, \geq 1$), can be solved for $K \leq \lceil \frac{\Delta}{m} \rceil$. The next sections will present an algorithm solving CCMC($n, m, K, \geq 1$) in the synchronous model described in Section 2, and a proof of it. As shown next, this algorithm is such $K = \lceil \frac{\Delta}{m} \rceil + 1$, and is consequently optimal with respect to the total number of colors.

Theorem 1. *Neither* CCMC($n, m, K, 1$)*, nor* CCMC($n, m, K, \geq 1$) *can be solved when* $K \leq \lceil \frac{\Delta}{m} \rceil$.

Proof Let us first show that there is no algorithm solving CCMC($n, m, K, 1$) when $K \leq \lceil \frac{\Delta}{m} \rceil$. To this end, let us consider a process p_ℓ, which has Δ neighbors (by the very definition of Δ, there is a such process). Let $\Delta = m \times x + y$, where $0 \leq y < m$. Hence, $x = \frac{\Delta - y}{m} = \lfloor \frac{\Delta}{m} \rfloor$ colors are needed to color $\Delta - y = m \times x$ processes. Moreover, if $y \neq 0$, one more color is needed to color the $y < m$ remaining processes. It follows that $\lceil \frac{\Delta}{m} \rceil$ is a lower bound to color the neighbors of p_ℓ. As p_ℓ cannot have the same color as any of its neighbors, it follows that at least $\lceil \frac{\Delta}{m} \rceil + 1$ are necessary to color $\{p_i\} \cup neighbors_i$, which proves the theorem for CCMC($n, m, K, \geq 1$).

Let us observe that an algorithm solving CCMC($n, m, K, 1$) can be obtained from an algorithm solving CCMC($n, m, K, \geq 1$) by associating with each p_i a single color of its set $colors_i$. Hence, any algorithm solving CCMC($n, m, \lceil \frac{\Delta}{m} \rceil, \geq 1$) can be used to solve CCMC($n, m, \lceil \frac{\Delta}{m} \rceil, 1$). As CCMC($n, m, \lceil \frac{\Delta}{m} \rceil, 1$) is impossible to solve, it follows that CCMC($n, m, \lceil \frac{\Delta}{m} \rceil, \geq 1$) is also impossible to solve. $\square_{Theorem\ 1}$

4.2 A necessary and sufficient condition for multicoloring

Let CCMC($n, m, \lceil \frac{\Delta}{m} \rceil + 1, > 1$) denote the problem CCMC($n, m, \lceil \frac{\Delta}{m} \rceil + 1, \geq 1$) where at least one node obtains more than one color.

Theorem 2. CCMC($n, m, \lceil \frac{\Delta}{m} \rceil + 1, > 1$) *can be solved on a tree of maximal degree* Δ*, if and only if*

$$\exists i : \lceil \frac{\Delta}{m} \rceil + 1 > \mathsf{max} \left(\left\{ \left\lceil \frac{\Delta_i}{m} \right\rceil \right\} \cup \left\{ \left\lfloor \frac{\Delta_j}{m} \right\rfloor \;\middle|\; p_j \in neighbors_i \right\} \right) + 1.$$

The proof of this theorem appears in Appendix A.

5 CCMC($n, m, K, \geq 1$) in a Tree Network: Algorithm

The algorithm presented in this section use as a skeleton a parallel traversal of a tree [21]. Such a traversal is implemented by control messages that visit all the processes, followed by a control flow that returns at the process that launched the tree traversal.

Algorithm 1 is a C2m-free algorithm that solves the CCMC($n, m, \lceil \frac{\Delta}{m} \rceil, \geq 1$) problem. It assumes that a single process initially receives an external message START(), which dynamically defines it as the root of the tree. This message and the fact that processes at distance smaller or equal to 2 do not have the same identity provide the initial asymmetry from which a deterministic coloring algorithm can be built. The reception of the message START() causes the receiving process (say p_r) to simulate the reception of a fictitious message COLOR(), which initiates the sequential traversal.

Messages The algorithm uses two types of messages, denoted COLOR() and TERM().

- The messages COLOR() implement a control flow visiting in parallel the processes of the tree from the root to the leaves. Each of them carries three values, denoted $sender$, cl_map, and max_cl.

- $sender$ is the identity of the sender of the message. If it is the first message COLOR() received by a process p_i, $sender$ defines the parent of p_i in the tree.
- cl_map is a dictionary data structure with one entry for each element in $neighbors_x \cup \{id_x\}$, where p_x is the sender of the message COLOR(). $cl\_map[id_x]$ is the set of colors currently assigned to the sender and, for each $id_j \in neighbor_x$, $cl\_map[id_j]$ is the set of colors that p_x proposes for p_j.
- max_cl is an integer defining the color domain used by the sender, namely the color set $\{0, 1, \ldots, (max\_cl - 1)\}$. Each child p_i of the message sender will use the color domain defined by $\mathsf{max}(max\_cl, \sigma_i)$ to propose colors to its own children (σ_i is defined below). Moreover, all the children of the sender will use the same slot span $\{0, 1, \ldots, (max\_cl - 1)\}$ to broadcast their messages. This ensures that their message broadcasts will be collision-free[5].

- The messages TERM() propagate the return of the control flow from the leaves to the root. Each message TERM() carries two values: the identity of the destination process (as this message is broadcast, this allows any receiver to know if the message is for it), and the identity of the sender.

Local variables Each process p_i manages the following local variables. The constant $\Delta_i = |neighbors_i|$ is the degree of p_i, while the constant $\sigma_i = \lceil \frac{\Delta_i}{m} \rceil + 1$ is the number of colors needed to color the star graph made up of p_i and its neighbors.

- $state_i$ (initialized to 0) is used by p_i to manage the progress of the tree traversal. Each process traverses five different states during the execution of the algorithm. States 1 and 3 are active states: a process in state 1 broadcasts a COLOR() message for its neighbors, while a process in state 3 broadcasts a message TERM() which has a meaning only for its parent. States 0 and 2 are waiting states in which a process listens on the broadcast channels but cannot send any message. Finally, state 4 identifies local termination.
- $parent_i$ stores the identity of the process p_j from which p_i receives a message COLOR() for the first time (hence p_j is the parent of p_i in the tree). The root p_r of the tree, defined by the reception of the external message START(), is the only process such that $parent_r = id_r$.
- $colored_i$ is a set containing the identities of the neighbors of p_i that have been colored.
- to_color_i is the set of neighbors to which p_i must propagate the coloring (network traversal).
- $color\_map_i[neighbors_i \cup \{id_i\}]$ is a dictionary data structure where p_i stores colors of its neighbors in $color\_map_i[neighbors_i]$, and its own colors in $color\_map_i[id_i]$; $colors_i$ is used as a synonym of $color\_map_i[id_i]$.
- max_cl_i defines both the color domain from which p_i can color its children, and the time slots (rounds) at which its children will be allowed to broadcast.
- $slot\_span_i$ is set to the value max_cl carried by the message COLOR() received by p_i from its parent. As this value is the same for all the children of its parent, they will use the same slot span to define the slots during which each child will be allowed to broadcast messages.

Initial state In its initial state ($state_i = 0$), a process p_i waits for a message COLOR(). As already indicated, a single process receives the external message START(), which defines it at the root process. It is assumed that $CLOCK = 0$ when a process receives this message. When it receives it, the corresponding process p_i simulates the reception of the message COLOR(id_i, cl_map, σ_i) where $cl\_map[id_i]$ defines its color, namely, $(CLOCK + 1)$ mod σ_i (lines 01-02). Hence, at round number 1, the root will send a message COLOR() to its children (lines 19-20).

[5] As we will see, conflicts are prevented by the message exchange pattern imposed by the algorithm.

Initialization: $\sigma_i = \lceil \frac{\Delta_i}{m} \rceil + 1$; $state_i \leftarrow 0$; $colors_i \leftarrow \emptyset$; $colors_i$ is a synonym of $color\_map_i[id_i]$.

(01) **when** START() **is received do** % a single process p_i receives this external message %
(02) p_i executes lines 04-25 as if it received the message COLOR(id_i, cl_map, σ_i)
where $cl\_map[id_i] = \{(CLOCK + 1) \bmod \sigma_i\}$.

(03) **when** COLOR$(sender, cl_map, max_cl)$ **is received do**
(04) **if** (first message COLOR() received)
(05) **then** $parent_i \leftarrow sender$; $color\_map_i[parent_i] \leftarrow cl\_map[sender]$;
(06) $colored_i \leftarrow \{sender\}$; $to\_color_i \leftarrow neighbors_i \setminus \{sender\}$;
(07) $color\_map_i[id_i] \leftarrow cl\_map[id_i]$; % Synonym of $colors_i$ %
(08) $max\_cl_i \leftarrow \max(max\_cl, \sigma_i)$; $slot\_span_i \leftarrow max\_cl$;
(09) **if** $(to\_color_i \neq \emptyset)$ % next lines: $tokens_i$ is a multiset %
(10) **then** $tokens_i \leftarrow$ { m tokens with color x, for each $x \in \big([0..(max\_cl_i - 1)] \setminus colors_i\big)$ }
$\setminus$ { one token with color z, for each $z \in color\_map_i[parent_i]$ };
(11) **while** $(|tokens_i| < |to\_colors_i|)$ **do**
(12) **if** $(|colors_i| > 1)$ **then let** $cl \in colors_i$; suppress cl from $colors_i$
(13) add m tokens colored cl to $tokens_i$
(14) **else let** cl be the maximal color in $color\_map_i[parent_i]$;
(15) add one token colored cl to $tokens_i$;
(16) $color\_map_i[parent_i] \leftarrow color\_map_i[parent_i] \setminus \{cl\}$
(17) **end if**
(18) **end while**;
(19) Extract $|to\_colors_i|$ non-empty non-intersecting multisets $tk[id]$ (where $id \in to\_color_i$)
from $tokens_i$ such that no $tk[id]$ contains several tokens with the the same color;
(20) **for each** $id \in to\_color_i$ **do** $color\_map_i[id] \leftarrow$ {colors of the tokens in $tk[id]$} **end for**;
(21) $state_i \leftarrow 1$ % p_i has children %
(22) **else** $state_i \leftarrow 3$ % p_i is a leaf %
(23) **end if**
(24) **else** $color\_map_i[id_i] \leftarrow color\_map_i[id_i] \cap cl\_map[id_i]$
(25) **end if**.

(26) **when** $\big((CLOCK \bmod slot\_span_i) \in colors_i) \wedge (state_i \in \{1, 3\})\big)$ **do**
(27) **case** $(state_i = 1)$ **then** broadcast COLOR$(id_i, color_map_i, max_cl_i)$; $state_i \leftarrow 2$
(28) $(state_i = 3)$ **then** broadcast TERM$(parent_i, id_i)$; $state_i \leftarrow 4$ % p_i's subtree is colored %
(29) **end case**.

(30) **when** TERM$(dest, id)$ **is received do**
(31) **if** $(dest \neq id_i)$ **then** discard the message (do not execute lines 25-28) **end if**;
(32) $colored_i \leftarrow colored_i \cup \{id\}$;
(33) **if** $(colored_i = neighbors_i)$
(34) **then if** $(parent_i = id_i)$ **then** the root p_i claims termination **else** $state_i \leftarrow 3$ **end if**
(35) **end if**.

Algorithm 1: C2m-free algorithm solving CCMC$(n, m, \lceil \frac{\Delta}{m} \rceil + 1, \geq 1)$ in tree networks (code for p_i)

Algorithm: reception of a message COLOR() When a process p_i receives a message COLOR() for the first time, it is visited by the network traversal, and must consequently (a) obtain an initial color set, and (b) propagate the the network traversal, if it has children. The processing by p_i of this first message COLOR$(sender, cl_map, max_cl)$ is done at lines 05-23. First, p_i saves the identity of its parent (the sender of the message) and its proposed color set (line 05), initializes $colored_i$ to $\{sender\}$, and to_color_i to its other neighbors (line 06). Then p_i obtains a color set proposal from the dictionary cl_map carried by the message (line 07), computes the value max_cl_i from which its color palette will be defined, and saves the value max_cl carried by the message COLOR() in the local variable $slot\_span_i$ (line 08). Let us remind that the value max_cl_i allows it to know the color domain used up to now, and the rounds at which it will be able to broadcast messages (during the execution of the algorithm) in a

collision-free way.

Then, the behavior of p_i depends on the value of to_color_i. If to_color_i is empty, p_i is a leaf, and there is no more process to color from it. Hence, p_i proceeds to state 3 (line 22).

If to_color_i is not empty, p_i has children. It has consequently to propose a set of colors for each of them, and save these proposals in its local dictionary $color\_map_i[neighbors_i]$. To this end, p_i computes first the domain of colors it can use, namely, the set $\{0, 1, \ldots, (max\_cl_i - 1)\}$, and considers that each of these colors c is represented by m tokens colored c. Then, it computes the multiset[6], denoted $tokens_i$, containing all the colored tokens it can use to build a color set proposal for each of its children (line 10). The multiset $tokens_i$ is initially made up of all possible colored tokens, from which are suppressed (a) all tokens associated with the colors of p_i itself, and, (b) one colored token for each color in $color\_map_i[parent_i]$ (this is because, from a coloring point of view, its parent was allocated one such colored token for each of its colors).

Then, p_i checks if it has enough colored tokens to allocate at least one colored token to each of its children (assigning thereby the color of the token to the corresponding child). If the predicate $|tokens_i| \geq |to\_color_i|$ is satisfied, p_i has enough colored tokens and can proceed to assign set of colors to its children (lines 19-20). Differently, if the predicate $|tokens_i| < |to\_color_i|$ is satisfied, p_i has more children than colored tokens. Hence, it must find more colored tokens. For that, if $colors_i$ (i.e., $color\_map_i[id_i]$) has more than one color, p_i suppresses one color from $colors_i$, adds the m associated colored tokens to the multiset $tokens_i$ (lines 12-13), and re-enters the "while" loop (line 11). If $colors_i$ has a single color, this color cannot be suppressed from $colors_i$. In this case, p_i considers the color set of its parent ($color\_map_i[parent_i]$), takes the maximal color of this set, suppresses it from $color\_map_i[parent_i]$, adds the associated colored token to the multiset $tokens_i$, and –as before– re-enters the "while" loop (line 15). Only one token colored cl is available because the $(m - 1)$ other tokens colored cl were already added into the multiset $tokens_i$ during its initialization at line 10.

As already said, when the predicate $|tokens_i| < |to\_color_i|$ (line 11) becomes false, $tokens_i$ contains enough colored tokens to assign to its children. This assignment is done at lines 19-20. Let $ch = |to\_color_i|$ (number of children of p_i); p_i extracts ch pairwise disjoint and non-empty subsets of the multiset $tokens_i$, and assigns each of them to a different neighbor. "Non-empty non-intersecting multisets" used at line 19 means that, if each of z multisets $tk[id_x]$ contains a token with the same color, this colored token appears at least z times in the multiset $tokens_i$.

If the message COLOR$(sender, cl_map, -)$ received by p_i is not the first one, it was sent by one of its children. In this case, p_i keeps in its color set $color\_map_i[id_i]$ ($colors_i$) only colors allowed by its child $sender$ (line 24). Hence, when p_i has received a message COLOR() from each of its children, its color set $colors_i$ has its final value.

Algorithm: broadcast of a message A process p_i is allowed to broadcast a message only at the rounds corresponding to a color it obtained (a color in $colors_i = color\_map_i[id_i]$ computed at lines 07, 12, and 24), provided that its current local state is 1 or 3 (line 26).

If $state_i = 1$, p_i received previously a message COLOR(), which entailed its initial coloring and a proposal to color its children (lines 09-21). In this case, p_i propagates the tree traversal by broadcasting a message COLOR() (line 27), which will provide each of its children with a coloring proposal. Process p_i then progresses to the local waiting state 2.

If $state_i = 3$, the coloring of the tree rooted at p_i is terminated. Process p_i consequently broadcasts the message TERM$(parent_i, id_i)$ to inform its parent of it. It also progresses from state 3 to state 4, which indicates its local termination (line 28).

[6]Differently from a set, a *multiset* (also called a *a bag*), can contain several times the same element. Hence, while $\{a, b, c\}$ and $\{a, b, a, c, c, c\}$ are the same set, they are different multisets.

Algorithm: reception of a message TERM() When a process p_i receives such a message it discards it if it is not the intended destination process (line 31). If the message is for it, p_i adds the sender identity to the set $colored_i$ (line 32). Finally, if $colored_i = neighbors_i$, p_i learns that the subtree rooted at it is colored (line 33). It follows that, if p_i is the root ($parent_i = i$), it learns that the algorithm terminated. Otherwise, it enters state 3, that will direct it to report to its parent the termination of the coloring of the subtree rooted at it.

Solving CCMC$(n, m, K, 1)$ in a tree Algorithm 1 can be easily modified to solve CCMC$(n, m, K, 1)$. When a process enters state 3 (at line 22 or line 34), it reduces $color\_map_i[id_i]$ (i.e., $colors_i$) to obtain a singleton.

6 CCMC$(n, m, K, \geq 1)$ in a Tree Network: Cost and Proof

The proof assumes $n > 1$. Let us remember that $colors_i$ and $color\_map_i[id_i]$ are the same local variable of p_i, and p_r denotes the dynamically defined root process.

Cost of the algorithm Each non-leaf process broadcasts one message COLOR(), and each non-root process broadcasts one message TERM(). Let x be the number of leaves. There are consequently $(2n - (x + 1))$ broadcasts. As $\Delta \leq x + 1$ ([7]), the number of broadcast is upper bounded by $2n - \Delta$.

Given an execution whose dynamically defined root is the process p_r, let d be the height of the corresponding tree. The root computes the colors defining the slots (rounds) at which its children can broadcast the messages COLOR() and TERM(). These colors span the interval $[0..\lceil \frac{\Delta_r}{m} \rceil]$, which means that the broadcasts of messages COLOR() by the processes at the first level of the tree span at most $\lceil \frac{\Delta_r}{m} \rceil + 1$ rounds. The same broadcast pattern occurs at each level of the tree. It follows that the visit of the tree by the messages COLOR() requires at most $d\lceil \frac{\Delta}{m} \rceil$ rounds. As the same occurs for the the messages TERM(), returning from the leaves to the root, it follows that the time complexity of the algorithm is $O(d\lceil \frac{\Delta}{m} \rceil)$.

Lemma 1. *Algorithm* 1 *is conflict-free.*

Proof The algorithm uses two types of messages: COLOR() and TERM(). We first show conflict-freedom for COLOR() messages (if a process broadcasts a message COLOR(), none of its neighbors is broadcasting any message in the same round). Let us first notice that a process p_i broadcasts at most one message COLOR(), and one message TERM() (this is due to the guard $state_i \in \{1, 3\}$, line 26, and the fact that the broadcast of a message makes its sender progress to the waiting state 2 or 4). Moreover, let us make the following observations.

- Observation 1: The first message sent by any node is of type COLOR() (line 27).
- Observation 2: Except for the root process, a message COLOR() is always broadcast by a process after it received a message COLOR() (which triggers the execution of lines 03-25).
- Observation 3: Except for leaf processes, a message TERM() is always broadcast by a process after it received a message TERM() from each of its children (lines 30-35 and line 28.).

Observations 1 and 2 imply that when the root process broadcasts its COLOR() message, none of its neighbors is broadcasting a message, and they all receive the root's COLOR() message without conflict. Let us now consider a process p_i, different from the root, which receives its first message COLOR$_k$() (from its parent p_k). Because there is no cycle in the communication graph (a tree), all the children of p_i

[7] Let p_i be the process that has Δ as degree. If p_i is the root of the tree, the tree contains at least Δ leaf processes. This is because each neighbor of p_i is either a leaf or the root of a subtree that has at least one leaf process. And if p_i is not the root of the tree, p_i possesses $\Delta - 1$ children, and the number of leaf processes is at least $\Delta - 1$ following a similar reasoning.

($neighbors_i \setminus \{p_k\}$) are in state 0, waiting for their COLOR() message. Moreover, due to Observations 1 and 2, they will receive from p_i their message COLOR() without conflict. After sending its COLOR() message, p_i's parent p_k remains in the waiting state 2 until it receives a TERM() message from all its children (lines 32-33), which include p_i. As a consequence, p_k is not broadcasting any message in the round in which it receives p_i's COLOR() message, which is consequently received without conflict by all its neighbors.

As far the messages TERM() are concerned we have the following. Initially, only a leaf process can broadcast a message, and when it does it, its parent is in the waiting state 2 (since it broadcast a message COLOR() at line 27 and it must receive messages TERM() to proceed to state 3). Hence a message TERM() broadcast by a leaf cannot entail conflict. Let us now consider a non-leaf process p_i. It follows from Observation 3 that p_i can broadcast a message TERM() only when its children are in state 4 (in which they cannot broadcast), and its parent (because it has not yet received a message TERM() from each of its children) is in the waiting state 2. Hence, we conclude that the broadcast of a message TERM() by a non-leaf process is conflict-free, which concludes the proof of the lemma. $\square_{Lemma\ 1}$

Definition A message COLOR$(sender, cl_map, max_cl)$ is *well-formed* if its content satisfies the following properties. Let $sender = id_i$.

M1 The keys of the dictionary data structure cl_map are the identities in $neighbors_i \cup \{id_i\}$.

M2 $\forall\ id \in (neighbors_i \cup \{id_i\}) :\ cl\_map[id] \neq \emptyset$.

M3 $\forall\ id \in neighbors_i :\ cl\_map[id] \cap cl\_map[id_i] = \emptyset$.

M4 $\forall c :\ |\{j : (id_j \in neighbors_i) \wedge (c \in cl\_map[id_j])\}| \leq m$.

M5 $1 < max\_cl \leq \lceil \frac{\Delta}{m} \rceil + 1$.

M6 $\forall\ id \in (neighbors_i \cup \{id_i\}) :\ cl\_map[id] \subseteq [0..max\_cl - 1]$.

Once established in Lemma 3, not all properties M1-M6 will be explicitly used in the lemmas that follow. They are used by induction to proceed from one well-formed message to another one.

Lemma 2. *If a message* COLOR$(sender, cl_map, max_cl)$ *received by a process $p_i \neq p_r$ is well-formed and entails the execution of lines* 05-23*, the* **while** *loop (lines* 11-18*) terminates, and, when p_i exits the loop, the sets $colors_i$ and $color\_map_i[parent_i]$ are not empty, and their intersection is empty.*

Proof Let us consider a process $p_i \neq p_r$ that receives a well-formed COLOR$_j(sender, cl_map, max_cl)$ message from p_j. Let us assume COLOR() causes p_i to start executing the lines 05-23, i.e., COLOR() is the first such message received by p_i. The body of the **while** loop contains two lines (lines 12 and 14) that select elements from two sets, $colors_i$ and $color\_map_i[parent_i]$ respectively.

Before discussing the termination of the **while** loop, we show that lines 12 and 14 are *well-defined*, i.e. the sets from which the elements are selected are non-empty. To this aim, we prove by induction that the following invariant holds in each iteration of the loop:

$$color\_map_i[parent_i] \neq \emptyset, \tag{1}$$

$$colors_i \neq \emptyset, \tag{2}$$

$$|tokens_i| = m \times max\_cl_i - m \times |colors_i| - |color\_map_i[parent_i]|. \tag{3}$$

Just before the loop (i.e., before line 11), Assertion (1) follows from the assignment to $color\_map_i[parent_i]$ at line 05 and the property M2 of COLOR$_j$() ($id_j = parent_i$). Assertion (2) also follows from M2 ($colors_i$ is synonym of $color\_map_i[id_i]$). Assertion (3) follows from M3, M6, and the initialization of max_cl_i at line 08.

Let us now assume that Assertion (1) holds at the start of a loop iteration (i.e., just before lines 12). There are two cases.

- If $|colors_i| > 1$, lines 14-16 are not executed, and consequently $color\_map_i[parent_i]$ is not modified. It follows from the induction assumption that Assertion (1) still holds.
- If $|colors_i| \leq 1$, we have the following. Because we are in the **while** loop, we have $|tokens_i| < |to\_colors_i|$, which, combined with Assertion (3), implies
$$|to\_colors_i| > m \times max\_cl_i - m \times |colors_i| - |color\_map_i[parent_i]|,$$
from which we derive
$$\begin{array}{rll} |color\_map_i[parent_i]| & > & m \times max\_cl_i - m \times |colors_i| - |to\_colors_i|, \\ & > & m \times max\_cl_i - m \times |colors_i| - (\Delta_i - 1) \quad \text{(because of line 06)}, \\ & > & m \times \sigma_i - m \times |colors_i| - (\Delta_i - 1) \quad \text{(because of line 08)}, \\ & > & m \times (\lceil \frac{\Delta_i}{m} \rceil + 1) - m \times |colors_i| - (\Delta_i - 1) \quad \text{(by definition)}, \\ & > & \Delta_i + m - m \times |colors_i| - (\Delta_i - 1) \quad \text{(arithmetic)}, \\ & > & m \times (1 - |colors_i|) + 1. \end{array}$$
Hence, because $|colors_i| \leq 1$, we obtain $|color\_map_i[parent_i]| > 1$, which means that p_i's local variable $color\_map_i[parent_i]$ contains at least two elements before the execution of line 12. Because only one color is removed from $color\_map_i[parent_i]$, this local variable remains non-empty after line 16, thus proving Assertion (1).

Let us now assume that both Assertion (2) and Assertion (3) hold at the start of a loop iteration (i.e., just before line 12). There are two cases.

- Case $|colors_i| > 1$. In this case we have: *(i)* one color is removed from $colors_i$, *(ii)* m colored tokens are added to $tokens_i$, and *(iii)* $color\_map_i[parent_i]$ remains unchanged. $|colors_i| > 1$ and *(i)* imply that Assertion (2) remains true; and *(i)* and *(ii)* mean that Assertion (3) is preserved.
- Case $|colors_i| \leq 1$. In this case we have: *(i)* one color is removed from $color\_map_i[parent_i]$, and one colored token added to $tokens_i$, and *(ii)* $colors_i$ stays unchanged. *(i)* implies that Assertion (3) remains true, and *(ii)* ensures Assertion (2) by assumption.

This concludes the proof that the three assertions (1)–(3) are a loop invariant. Hence, Assertion (1) and Assertion (2) imply that lines 12 and 14 are well-defined.

Let us now observe that, in each iteration of the loop, new colored tokens are added to $tokens_i$, and thus $|tokens_i|$ is strictly increasing. Because $|to\_color_i|$ remains unchanged, the condition $|tokens_i| < |to\_color_i|$ necessarily becomes false at some point, which proves that the loop terminates.

Just after the loop, the invariant is still true. In particular Assertion (1) and Assertion (2) show that both the sets $colors_i$ and $color\_map_i[parent_i]$ are not empty when p_i exits the **while** loop.

Finally, due to to the fact that the message $\text{COLOR}_j()$ is well-formed, it follows from M3 that we have $colors_i \cap color\_map_i[parent_i] = \emptyset$ after line 07. As colors are added neither to $colors_i$, nor to $color\_map_i[parent_i]$ in the loop, their intersection remains empty, which concludes the proof of the lemma. $\square_{Lemma\ 2}$

Lemma 3. *All messages* COLOR() *broadcast at line* 27 *are well-formed.*

Proof To broadcast a message COLOR(), a process p_i must be in local state 1 (line 27). This means that p_i executed line 21, and consequently previously received a message $\text{COLOR}(sender, cl\_map, max\_cl)$ that caused p_i to execute lines 05-23.

Let us first assume that COLOR() is well-formed. It then follows from Lemma 2 that p_i exits the while loop, and each of $colors_i$ and $color\_map_i[parent_i]$ is not empty (A), and they have an empty intersection (B). When considering the message $\text{COLOR}(id_i, color\_map_i, max\_cl_i)$ broadcast by p_i we have the following.

- M1 follows from the fact that the entries of the dictionary data structure created by p_i are: $color\_map_i[parent_i]$ (line 05), $color\_map_i[id_i]$ (line 07), and $color\_map_i[id]$ for each $id \in to\_colors_i = neighbors_i \setminus \{parent_i\}$ (lines 06 and 20), and the observation that no entry is ever removed from $color\_map_i$ is the rest of the code.
- M2 follows from (A) for $color\_map_i[parent_i]$ and $color\_map_i[id_i]$, from line 19 for the identities in $to\_colors_i = neighbors_i \setminus \{parent_i\}$ (due to $|tokens_i| \geq |to\_colors_i|$ when line 19 is executed, and the non-intersection requirement of the $tk[id]$ sets, no $tk[id]$ is empty), and from the observation that $color\_map_i$ is not modified between the end of line 20 and the broadcast of line 27. This last claim is derived from the fact that $color\_map_i$ is only modified when messages are received, and that neither p_i's parent nor p_i's children are in states that allow them to send messages while p_i is transitioning from line 20 to line 27.
- Similarly M3 follows
  - for $id = parent_i$: from (B) and the fact that $color\_map_i[parent_i]$ never increases,
  - for $id \in to\_color_i = neighbors_i \setminus \{parent_i\}$: from the fact that, due to lines 10 and 12, at line 19 $tokens_i$ contains no token whose color belongs to $colors_i$, from which we have $tk[id] \cap color\_map_i[id_i] = \emptyset$ for any $id \in to\_color_i$.
- M4 follows from the construction of $tokens_i$. This construction ensures that, for any color c, $tokens_i$ contains at most m tokens with color c (line 10, 13, and 15).
- M5 is an immediate consequence of the assignment $max\_cl_i \leftarrow \mathsf{max}(max\_cl, \sigma_i)$ at line 15.
- M6 follows from the following observations:
  - for $id \in \{id_i, parent_i\}$: from $max\_cl \leq max\_cl_i$ (line 08) and the fact that the message COLOR() received by p_i is well-formed (hence $color\_map_i[id_i] \cup color\_map_i[parent_i] \subseteq [0..(max\_cl - 1)]$),
  - for $id \in to\_color_i = neighbors_i \setminus \{parent_i\}$: from the fact that $tokens_i$ contains only tokens whose color is in $[0..(max\_cl_i - 1)]$ (line 10).

The previous reasoning showed that, if a process receives a well-formed message COLOR(), executes lines 05-23 and line 27, the message COLOR$(id_i, color_map_i, max_cl_i)$ it will broadcast at this line is well-formed. Hence, to show that all messages broadcast at line 27 are well-formed, it only remains to show that the message COLOR$(id_r, color_map_r, max_cl_r)$ broadcast by the root p_r is well-formed. Let us remember that $neighbors_r$ is a constant defined by the structure of the tree, and $parent_r = id_r \notin neighbors_r$.

Let us notice that the message COLOR(id, cl_map, max_cl), that p_r sends to itself at line 02,is not well-formed. This is because, $cl\_map[id]$ is not defined for $id \in neighbors_i$. When p_r receives this message we have the following after line 10:

$$|tokens_r| = m \times |\sigma_r| - m = m \times (|\sigma_r| - 1) = m\lceil \tfrac{\Delta_r}{m} \rceil \geq \Delta_r,$$

from which we conclude $|tokens_r| \geq \Delta_r = |to\_colors_r| = |neighbors_i|$. Hence, p_r does not execute the loop body, and proceeds to lines 19-20 where it defines the entries $color\_map_r[id]$ for $id \in to\_colors_r = neighbors_r$. A reasoning similar to the previous one shows that the message COLOR$(id_r, color_map_r, max_cl_r)$ broadcast by p_r at line 27 satisfies the properties M1-M6, and is consequently well-formed. (The difference with the previous reasoning lies in the definition of the set to_colors_i which is equal to $neighbors_i \setminus \{parent_i\}$ for $p_i \neq p_r$, and equal to $neighbors_r$ for p_r.)

$\square_{Lemma\ 3}$

Lemma 4. *If a process p_i computes a color set ($colors_i$), this set is not empty.*

Proof Let us first observe that, if a process $p_i \neq p_r$ receives a message COLOR$(-, cl_map, -)$, the previous lemma means that this message is well-formed, and due to property M2, its field $cl\_map[id_i]$ is not empty, from which follows that the initial assignment of a value to $color\_map_i[id_i] \equiv colors_i$ is a non-empty set. Let us also observe, that, even if it is not well-formed the message COLOR$(-, cl_map, -)$ received by the root satisfies this property. Hence, any process that receives a message COLOR() assigns first a non-empty value to $color\_map_i[id_i] \equiv colors_i$.

Subsequently, a color can only be suppressed from $color\_map_i[id_i] \equiv colors_i$ at line 24 when p_i receives a message COLOR() from one of its children. If p_i is a leaf, it has no children, and consequently never executes line 24. So, let us assume that p_i is not a leaf and receives a message COLOR$(id_j, cl_map, -)$ from one of its children p_j. In this case p_i previously broadcast at line 27 a message COLOR$(id_i, color_map_i, -)$ that was received by p_j and this message is well-formed (Lemma 3).

A color c that is suppressed at line 24 when p_i processes COLOR$(id_j, cl_map, -)$ is such that $c \in colors_i$ and $c \notin cl\_map[id_i]$. $cl\_map[id_i]$ can be traced back to the local variable $color\_map_j[id_i]$ used by p_j to broadcast COLOR() at line 27. Tracing the control flow further back, $color\_map_j[id_i]$ was initialized by p_j to $color\_map_i[id_i]$ (line 05) when p_j received the well-formed message COLOR() from p_i.When processing COLOR() received from p_i, process p_j can suppress colors from $color\_map_j[id_i]$ only at line 16, where it suppresses colors starting from the greatest remaining color. We have the following.

- If p_i is not the root, the message COLOR() it received was well-formed (Lemma 3). In this case, it follows from the proof of Lemma 2 that it always remains at least one color in $color\_map_j[id_i]$.
- If $p_i = p_r$, its set $colors_r$ is a singleton (it "received" COLOR$(id_r, cl_map_r, -)$ where cl_map_r has a single entry, namely $cl\_map_r[id_r] = \{1\}$). When p_j computes $tokens_j$ (line 10) we have $|tokens_j| = m \times \mathsf{max}(|\sigma_r|, |\sigma_j|) - m = m\lceil \frac{\mathsf{max}(\Delta_r, \Delta_j)}{m} \rceil \geq \mathsf{max}(\Delta_r, \Delta_j) \geq \Delta_j = |to\_colors_j|$, from which follows that $|tokens_j| \geq |to\_colors_j| = |neighbors_j| - 1$. Hence, p_j does not execute the loop, and consequently does not modify $color\_map_j[id_r]$.

Consequently, the smallest color of $colors_i \equiv color\_map_i[id_i]$ is never withdrawn from $color\_map_j[id_i]$. It follows that, at line 24, p_i never withdraws its smallest color from the set $color\_map_i[id_i]$. $\square_{Lemma\ 4}$

Lemma 5. *If p_i and p_j are neighbors $colors_i \cap colors_j = \emptyset$.*

Proof As all color sets are initialized to $\emptyset$, the property is initially true. We show that, if a process receives a message COLOR(), the property remains true. As TERM() messages do not modify the coloring—lines 30-35—they do not need to be considered.

Let us consider two neighbor processes p_i and p_j, which computes their color sets (if none or only one of p_i and p_j computes its color set, the lemma is trivially satisfied). As the network is a tree, one of them is the parent of the other. Let p_i be the parent of p_j.

Process p_i broadcast a message COLOR$(-, cl_map, -)$ at line 27 in which the set $cl\_map[id_j]$ is $color\_map_i[id_j]$, as computed at line 19. If this message is received by p_j, this set will in turn be assigned to $color\_map_i[id_j]$ at p_j. As this message is well-formed (Lemma 3), we therefore have $color\_map_i[id_i] \cap color\_map_j[id_j] = \emptyset$ (Property M3 of a well-formed message). Then, while p_i can be directed to suppress colors from $color\_map_i[id_i]$ at line 24, it never adds a color to this set. The same is true for p_j and $color\_map_j[id_j]$. It follows that the predicate $color\_map_i[id_i] \cap color\_map_j[id_j] = \emptyset$ can never be invalidated. $\square_{Lemma\ 5}$

Lemma 6. $\forall i, \forall c : |\{j : j \in neighbors_i \wedge c \in colors_j\}| \leq m$.

Proof The property is initially true. We show that it remains true when processes receive messages.

Let us consider a process p_i that broadcasts a message COLOR(). Due to the fact that such messages are broadcast only at line 27, it follows from Lemma 3 that the message COLOR$(id_i, cl_map, -)$ broadcast by p_i is well-formed. Hence it satisfies property M4. When processing this message

A each child p_j of p_i adopts $cl\_map[id_j]$ as its initial color set and assigns it to $color\_map_j[id_j]$;

B p_i's parent p_k uses $cl\_map[id_k]$ to update $color\_map_k[id_k]$ at line 24 such that $color\_map_k[id_k] \subseteq cl\_map[id_k]$.

(A), (B), and M4 imply that just after p_i's neighbors have processed p_i's message, the lemma holds. As already seen in the proof of other lemmas, $color\_map_j[id_j]$ may subsequently decrease, but never increases: colors can be suppressed from $color\_map_j[id_j]$ (line 24) but never added to it. And the same is true at p_i for its set of colors $color\_map_i[id_i]$, and at its parent p_k for $color\_map_k[id_k]$. It then follows that $|\{j : j \in neighbors_i \wedge c \in colors_j\}| \leq m$ throughout the execution of the algorithm, which concludes the proof of the lemma. $\square_{Lemma\ 6}$

Lemma 7. *Algorithm* 1 *is collision-free.*

Proof We have to show that no process can have more than m of its neighbors that broadcast during the same round. Initially, all processes are in state 0. Let us consider a process p_i and assume that one of its neighbors p_j is broadcasting a message. Let us further assume that this message is of type COLOR().

- If p_j is p_i's parent, p_j's COLOR() message is the first message received by p_i, and both p_i and its children (p_i's remaining neighbors) are in state 0, and hence silent. There is no collision at p_i.
- If p_j is one of p_i's children, the value $slot\_span_j$ used by p_j at line 26 is equal to max_cl contained in the message COLOR$(-, -, max_cl)$ first received by p_j from p_i. Because of Lemma 3, this message is well-formed, and consequently satisfies property M6. Any other child p_ℓ of p_i broadcasting during this round will have received the same first message, and will therefore be using the same $slot\_span_\ell = max\_cl$ value. It follows from Property M6, the assignment of line 07 executed by any child p_ℓ (of p_i) that received the message, and the fact that its set $colors_\ell$ can only decrease after being first assigned, that $colors_\ell \subseteq [0..slot\_span_\ell - 1]$ for any child p_ℓ of p_i (C).

  Lemma 6, Property (C), and the $CLOCK$-based predicate defining the rounds at which a process is allowed to broadcast (line 26), imply that at most m children of p_i can broadcast during the same round. If p_i has a parent p_k (i.e. p_i is not the root), both p_i and p_k are in state 2, and hence p_k is silent, proving the lemma. If p_i is the root, all its neighbors are its children, and the lemma also holds.

The same reasoning applies to the messages TERM() broadcast by the children of p_i and its parent.

$\square_{Lemma\ 7}$

Lemma 8. *Each process computes a set of colors, and the root process knows when coloring is terminated.*

Proof Let us first observe that, due to Lemmas 2 and 3, no process $p_i \neq p_r$ can loop forever inside the **while** loop (lines 11-18), when it receives its first message COLOR(). The same was proved for the root p_r at the end of the proof of Lemma 4. Moreover, a process cannot block at line 24 when it receives other messages COLOR() (one from each of its children). Hence, no reception of a message COLOR() can prevent processes from terminating the processing of the message. The same is trivially true for the processing of a message TERM().

Let us first show that each process obtains a non-empty set of colors. To this end, we show that each non-leaf process broadcasts a message COLOR().

- When the root process p_r receives the external message START(), it "simulates the sending to itself" of the message COLOR$(id_r, color_map_r, \sigma_r)$, where the dictionary data structure $color\_map_r$ has a single element, namely, $color\_map_r[id_r] = \{1\}$. The root p_r executes consequently the

lines 25-23, during which it obtains a color ($color\_map_r[id_r] = \{1\}$, line 07), and computes a set of proposed colors $color\_map_r[id_j]$ for each of its children p_j (lines 19-20). It then progresses to the local non-waiting state 1 (line 21). Hence, during the first round, it broadcasts to its neighbors the message COLOR$(id_r, color_map_r, \sigma_r)$.Because the algorithm is conflict- and collision-free (Lemmas 1 and 5), this message is received by all the root's neighbors.

- Let us now consider a process p_i that receives a message COLOR$(sender, color_map, max_cl)$ for the first time. It follows from Lemma 4 that p_i starts computing a non-empty set $colors_i$ and enters the waiting state 1 (line 21). Finally, as $colors_i \subseteq [0..slot\_span_i - 1]$, and $CLOCK$ never stops increasing, the predicate of line 26 is eventually satisfied. It follows p_i broadcasts the messageCOLOR$(id_i, color_map_i, max_cl_i)$. As above all of p_i's neighbors will receive this message.

  It follows that COLOR() messages flood the tree from the root to the leaves.

  Moreover, when a process p_i has received a message COLOR() from each of its neighbors (children and parent), it has obtained the final value of its color set $color\_map_i[id_i] = colors_i$. Due to lemma 4, this set is not empty, which concludes the first part of the proof.

Let us now show that the root learns coloring termination. This relies on the messages TERM(). As previously, due to Lemma 1 and Lemma 7, these messages entail neither message conflicts nor message collisions.

Let us observe that each leaf process enters the non-waiting state 3. When the predicate of line 26 is satisfied at a leaf p_ℓ (this inevitably occurs), this process broadcasts the message TERM() to its parent p_i. Then, when p_i has received a message TERM() from each of its children, it broadcasts TERM() to its own parent. This sequence repeats itself on each path from a leaf to the root. When the root has received a message TERM() from each of its children, it learns termination (line 34), which concludes the proof of the lemma. $\square_{Lemma\ 8}$

Lemma 9. $|\bigcup_{1\leq i\leq n} colors_i| = \lceil \frac{\Delta}{m} \rceil + 1$.

Proof Let $p_r, p_a, \cdots, p_\ell$ be a path in the tree starting at the root p_r and ending at a leaf p_ℓ. It follows from

- the content of the parameter max_cl of the messages COLOR$(sender, cl_map, max_cl)$ broadcast along this path of the tree (broadcast at line 27 and received at line 03), and
- the assignment of $\mathsf{max}(max\_cl, \sigma_i)$ to max_cl_i at line 08,

that $max\_cl_\ell = \mathsf{max}(\sigma_r, \sigma_a, \cdots, \sigma_\ell)$. Let $p_{\ell 1}$, ..., $p_{\ell x}$ be the set of leaves of the tree. It follows that $\mathsf{max}(max\_cl_{\ell 1}, \cdots, max\_cl_{\ell x}) = \mathsf{max}(\sigma_1, \cdots, \sigma_n)$, i.e., the value max_cl carried by any message is $\leq \lceil \frac{\Delta}{m} \rceil + 1$.

The fact that a process p_i uses only colors in $[0..(max\_cl_i - 1)]$, combined with Theorem 1 implies the lemma. The algorithm is consequently optimal with respect to the number of colors. $\square_{Lemma\ 9}$

Theorem 3. *Let $K = \lceil \frac{\Delta}{m} \rceil + 1$. Algorithm 1 is a* C2$m$-free *algorithm, which solves* CCMC$(n, m, K, \geq 1)$ *in tree networks. Moreover, it is optimal with respect to the value of K.*

Proof The proof that Algorithm 1 is C2m-free follows from Lemma 1 and Lemma 7. The proof that it satisfies the Conflict-freedom, Collision-freedom, and Efficiency properties defining the CCMC$(n, m, K, \geq 1)$ problem follows from Lemmas 2-6, and Lemma 8. The proof of its optimality with respect to K follows from Lemma 9. $\square_{Theorem\ 3}$

7 Conclusion

The paper first introduced a new vertex coloring problem (called CCMC), in which a process may be assigned several colors in such a way that no two neighbors share colors, and for any color c, at most m neighbors of any vertex share the color c. This coloring problem is particularly suited to assign rounds (slots) to processes (nodes) in broadcast/receive synchronous communication systems with communication or local memory constraints. Then, the paper presented a distributed algorithm which solve this vertex coloring problem for tree networks in a round-based programming model with conflicts and (multi-frequency) collisions. This algorithm is optimal with respect to the total number of colors that can be used, namely it uses only $K = \lceil \frac{\Delta}{m} \rceil + 1$ different colors, where Δ is the maximal degree of the graph.

It is possible to easily modify the coloring problem CCMC to express constraints capturing specific broadcast/receive communication systems. As an example, suppressing the conflict-freedom constraint and weakening the collision-freedom constraint into

$$\forall i, \forall c : |\{j : (id_j \in neighbors_i \cup \{id_i\}) \wedge (c \in colors_j)\}| \leq m, \tag{4}$$

captures bi-directional communication structures encountered in some practical systems in which nodes may send and receive on distinct channels during the same round. Interestingly, solving the coloring problem captured by (4) is equivalent to solving distance-2 coloring in the sense that a purely local procedure (i.e., a procedure involving no communication between nodes) executed on each node can transform a classical distance-2 coloring into a multi-coloring satisfying (4). More precisely, assuming a coloring $\mathsf{col} : V \mapsto [0..(K * m) - 1]$ providing a distance-2 coloring with $K * m$ colors on a graph $G = (V, E)$, it is easy to show that the coloring (with one color per vertex)

$$\begin{array}{rccl} \mathsf{col}' : & V & \mapsto & [0 \,..\, K-1] \\ & x & \rightarrow & \mathsf{col}(x) \bmod K, \end{array} \tag{5}$$

fulfills (4) on G ([8]). Since the distance-2 problem with $K*m$ colors is captured by CCMC$(n, 1, K*m, 1)$ (as discussed in Section 3), the proposed algorithm can also solve the coloring condition captured by (4) on trees in our computing model.

Moreover, from an algorithmic point of view, the proposed algorithm is versatile, making it an attractive starting point to address other related problems. For instance, in an heterogeneous network, lines 19-20 could be modified to take into account additional constraints arising from the capacities of individual nodes, such as their ability to use only certain frequencies.

Last but not least, a major challenge for future work consists in solving the CCMC problem in general graphs. The new difficulty is then to take into account cycles.

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A Proof of Theorem 2

Lemma 10. $K' > \left\lfloor \frac{\Delta_j}{m} \right\rfloor \iff K' \times m > \Delta_j$.

Proof From $K' > \lfloor \frac{\Delta_j}{m} \rfloor$ we can derive the following. Because K' is an integer, we have $K' \geq \lfloor \frac{\Delta_j}{m} \rfloor + 1$. Because $\lfloor x \rfloor > x - 1$, we obtain $K' > \frac{\Delta_j}{m} - 1 + 1$, from which we conclude $K' \times m > \Delta_j$. Conversely, if we have $K' \times m > \Delta_j$, then $K' > \frac{\Delta_j}{m}$, and, because $x \geq \lfloor x \rfloor$, we obtain $K' > \frac{\Delta_j}{m} \geq \lfloor \frac{\Delta_j}{m} \rfloor$, which concludes the proof of the lemma. $\square_{Lemma\ 10}$

Theorem 2 Let $K = \lceil \frac{\Delta}{m} \rceil + 1$. CCMC$(n, m, K, > 1)$ can be solved on a tree of maximal degree Δ, if and only if

$$\exists i : K > \mathsf{max}\left(\left\{ \left\lceil \frac{\Delta_i}{m} \right\rceil \right\} \cup \left\{ \left\lfloor \frac{\Delta_j}{m} \right\rfloor \;\middle|\; p_j \in neighbors_i \right\} \right) + 1.$$

Proof The terms "process" and "vertex" are considered here as synonyms. To simplify notation, we consider in the following that $id_i = i$. Let us first notice that if follows from its definition that $K \geq 2$.

Proof of the "if direction".
The proof of this direction consists in a sequential algorithm that associates two colors to a process p_i whose position in the tree satisfies the previous predicate.

Algorithm 2 is a sequential algorithm solving CCMC$(n, m, \lceil \frac{\Delta}{m} \rceil + 1, 1)$. Using the control flow defined by a simple depth-first tree traversal algorithm, it takes two input parameters, a process p_j, and its color. Then, assuming a coloring of both p_j and its parent, it recursively colors the vertices of the tree rooted at p_j. The initial call is DF_MColoring$(i, 0)$ where p_i is a vertex satisfying the predicate stated in the theorem, and 0 the color assigned to it. The function parent_color(j) returns the color of the parent of p_j if $p_j \neq p_i$ and returns no value if $p_j = p_i$. Let us notice that, except for $p_j = p_i$, parent_color(j) is called only after the parent of p_j obtained a color.

```
procedure DF_MColoring(j, c) is
(01)  color_i ← c;
(02)  if |neighbors_j| > 1 then
(03)     tokens ← { m colored tokens for each color in {0, 1, ··· , (K − 1)} \ {color_j, parent_color(j)} }
                  ∪ { (m − 1) tokens with color parent_color(j)};
(04)     for each p_k ∈ (neighbors_j \ {parent_j}) do   %this loop is executed (Δ_j − 1) times %
(05)        token ← a smallest token in tokens; suppress token from tokens;
(06)        DF_MColoring(k, color of token)
(07)     end for
(08)  end if.
```

Algorithm 2: Sequential multi-coloring of a tree with a depth-first traversal algorithm (code for p_i)

A call to DF_MColoring(j, c) works as follows. First, color c is assigned to p_j. If p_j has a single neighbor (its parent, which issued this call), the current procedure call terminates. Otherwise, the current invocation computes the multiset of colored tokens which includes (a) m identical tokens for each possible color, except the colors of p_j and its parent, and (b) $(m-1)$ identical tokens with the color of $parent_j$ (line 03). Let us notice that all the colored tokens are ordered by their color number, hence the notion of a "token with a smallest color" is well-defined. Then, for each of p_j's neighbor p_k (except its parent), taken one after the other (line 04), a token with a smallest color is selected for p_k (which will inherit the corresponding color) and withdrawn from the multiset $tokens$ (hence, this token can no longer be used to associate a color to another neighbor of p_j, line 04). Due to line 03, the multiset $tokens$ used to assign colors to p_j's neighbors, is such that $|tokens| = (K-2) \times m + (m-1) = (K-1) \times m - 1$ (Property P1). (The factor $(K-2)$ comes from the fact the colors are in the color set $\{0, 1, \cdots, (K-1)\} \setminus \{color_j, \text{parent\_color}(j)\}$. The term $(m-1)$ comes from the fact that $color_i = 0$ is already used once for the neighbor p_i.) It follows that the loop is well-defined. The subtree rooted at p_k is then depth-first recursively colored (line 05).

It follows that (a) no two neighbors can be assigned the same color (line 03), (b) each process is assigned a color as small as possible (line 04), and (c) at most m neighbors of a process can be assigned the same color (lines 02 and 05).

We now show that, given the previous coloring, it is possible to assign (at least) one more color to (at least) the root p_i. The set of K colors used in Algorithm 2 is the set $\{0, 1, \cdots, (K-1)\}$. Let us consider any vertex p_j, which is a neighbor of p_i. Due to the property stated in the theorem, we have $K > \lfloor \frac{\Delta_j}{m} \rfloor + 1$, which, due to Lemma 10 translates as $(K-1) \times m > \Delta_j$ (P2).

As $color_i = 0$ and $K \geq 2$, we have $color_i \neq K-1$. Moreover, we also have $color_j \neq K-1$. This follows from the assumption $K > \lceil \frac{\Delta_i}{m} \rceil + 1$, and the fact that, to color p_i and its neighbors, Algorithm 2

uses $\lceil \frac{\Delta_i}{m} \rceil + 1 \leq K - 1$ colors, namely the color set $\{0, \cdots, (K-2)\}$.

When executing DF_MColoring(j, c) for a neighbor p_j of p_i, Algorithm 2 executes $(\Delta_j - 1)$ times the body of the "for" loop (lines 04-06), (once for each neighbor of p_j, except p_i, which has already been assigned a color). It follows from (P2) that $(K-1) \times m - 1 > \Delta_j - 1$. Combined with (P1) we obtain $|tokens| > \Delta_j - 1$, from which we conclude that $tokens \neq \emptyset$ is always true. It is consequently a loop invariant in each call related to a neighbor p_j of p_i. It follows that $tokens$ always contains a colored token with the highest color, namely $(K-1)$. This color can consequently be assigned to p_i, in addition of color 0, without violating the conflict-freedom, m-collision-freedom, and efficiency defining the CCMC problem solved by Algorithm 2. This concludes the proof of the "if" part of the theorem.

Proof of the "only if direction".
In the following we use the following notations, where M is a multi-set.

- $|M|$ is the size of M (in the following all sets and multisets are finite),
- $set(M)$ is the underlying set of M, the set of elements present at least once in M,
- $\mathbf{1}_M(x)$ is the multiplicity of an element x in M. By construction we have

$$\mathbf{1}_M(x) \geq 1 \Longleftrightarrow x \in set(M) \quad \text{and} \quad |M| = \sum_{x \in set(M)} \mathbf{1}_M(x). \tag{6}$$

If A and B are two multisets, $A \uplus B$ is the multiset union of A and B. In particular we have:

$$|A \uplus B| = |A| + |B| \quad \text{and} \quad \mathbf{1}_{A \uplus B}(x) = \mathbf{1}_A(x) + \mathbf{1}_B(x). \tag{7}$$

We consider a set S as a special case of a multi-set in which all elements of S have a multiplicity of 1: $x \in S \Longleftrightarrow \mathbf{1}_S(x) = 1$.

Let us assume that CCMC$(n, m, K, > 1)$ can be solved on a tree, such that at least one process, i.e. p_i, is allocated more than one color:

$$|colors_i| \geq 2. \tag{8}$$

For ease of exposition, and without loss of generality, we assume all other processes are allocated only one color:

$$\forall j \neq i : |colors_j| = 1. \tag{9}$$

Let $C_{neighbors_i}$ denote the multiset of colored tokens allocated to the neighbors of p_i:

$$C_{neighbors_i} = \biguplus_{p_j \in neighbors_i} colors_j. \tag{10}$$

From (9) and (10) we derive (by way of (7))

$$|C_{neighbors_i}| = | \biguplus_{p_j \in neighbors_i} colors_j| = \sum_{p_j \in neighbors_i} |colors_j| = |neighbors_i| = \Delta_i. \tag{11}$$

This means that Δ_i colored tokens are needed to color the neighbors of p_i.
Because the coloring solves CCMC$(n, m, K, > 1)$, m-*Collision-freedom* means that

$$\forall c \in set(C_{neighbors_i}) : \mathbf{1}_{C_{neighbors_i}}(c) \leq m. \tag{12}$$

Using (12) in (6) applied to $C_{neighbors_i}$ gives us

$$|C_{neighbors_i}| = \sum_{c \in set(C_{neighbors_i})} \mathbf{1}_{C_{neighbors_i}}(c) \leq |set(C_{neighbors_i})| \times m, \tag{13}$$

which yields, with (11) (required number of colors for p_i's neighbors):

$$|set(C_{neighbors_i})| \geq \frac{|C_{neighbors_i}|}{m} \geq \frac{\Delta_i}{m}. \tag{14}$$

Because $|set(C_{neighbors_i})|$ is an integer, (14) implies that

$$|set(C_{neighbors_i})| \geq \left\lceil \frac{\Delta_i}{m} \right\rceil. \tag{15}$$

Because the coloring solves CCMC($n, m, K, > 1$), it respects *Conflict-freedom*, implying that

$$set(C_{neighbors_i}) \cap colors_i = \emptyset, \tag{16}$$

and hence

$$\begin{aligned} |set(C_{neighbors_i}) \cup colors_i| &= |set(C_{neighbors_i})| + |colors_i|, && (17)\\ &\geq |set(C_{neighbors_i})| + 2 && \text{using (8)}, \quad (18)\\ &\geq \left\lceil \frac{\Delta_i}{m} \right\rceil + 2 > \left\lceil \frac{\Delta_i}{m} \right\rceil + 1. && \text{using (15)} \quad (19) \end{aligned}$$

By definition $K \geq |set(C_{neighbors_i}) \cup colors_i|$, which yields

$$K > \left\lceil \frac{\Delta_i}{m} \right\rceil + 1 \tag{20}$$

which concludes the first part of the proof on the "only if" direction.
Let us now turn to the neighbors of p_i. For $p_j \in neighbors_i$ we consider similarly to p_i the set of colored tokens allocated to p_j's neighbors (which include p_i):

$$C_{neighbors_j} = \biguplus_{p_k \in neighbors_j} colors_k. \tag{21}$$

Hence (as for p_i) we have:

$$|C_{neighbors_j}| = \sum_{p_k \in neighbors_j} |colors_k|. \tag{22}$$

Contrary to p_i however, all of p_j's neighbors do not have only one color allocated: p_i has at least two, by assumption. This yields

$$\begin{aligned} |C_{neighbors_j}| &= |colors_i| + \sum_{p_k \in neighbors_j \setminus \{p_i\}} |colors_k|, && (23)\\ &\geq 2 + |neighbors_j - 1| \times 1 && \text{using (8) and (9)}, \quad (24)\\ &\geq 2 + \Delta_j - 1 \geq \Delta_j + 1. && (25) \end{aligned}$$

As for p_i, m-*Collision-freedom* means that

$$|C_{neighbors_j}| \leq |set(C_{neighbors_j})| \times m. \tag{26}$$

(25) and (26) yield

$$|set(C_{neighbors_j})| \times m \geq \Delta_j + 1. \tag{27}$$

As for p_i we have (*Conflict-freedom*)

$$set(C_{neighbors_j}) \cap colors_j = \emptyset, \tag{28}$$

leading to

$$\begin{aligned} K &\geq |set(C_{neighbors_j}) \cup colors_j| && \text{by definition,} && (29)\\ &\geq |set(C_{neighbors_j})| + |colors_j| && \text{because of (28),} && (30)\\ &\geq |set(C_{neighbors_j})| + 1 && \text{because of (9),} && (31)\\ K - 1 &\geq |set(C_{neighbors_j})|. && && (32) \end{aligned}$$

Injecting (32) into (27) gives us

$$\begin{aligned} (K-1) \times m &\geq \Delta_j + 1, && && (33)\\ (K-1) \times m &> \Delta_j, && && (34)\\ K - 1 &> \frac{\Delta_j}{m} \geq \left\lfloor \frac{\Delta_j}{m} \right\rfloor && \text{because of Lemma 10,} && (35)\\ K &> \left\lfloor \frac{\Delta_j}{m} \right\rfloor + 1. && && (36) \end{aligned}$$

(36) concludes the proof of necessary condition to solve CCMC$(n, m, K, > 1)$. $\square_{Theorem\ 2}$

B Defining the slots of the upper layer programming level

When the root process claims termination, the other processes are in their local state 4, but cannot exploit the multi-coloring assignment. As indicated in the paragraph "Using the colors" (just before Section 4), to do that, they need to know the value $K = \lceil \frac{\Delta}{m} \rceil + 1$.

The knowledge of K can be brought to the root process by the messages TERM(), and then disseminated from the root to all the processes. Algorithm 1 is slightly modified and enriched with Algorithm 3 to allow all processes to know the value of K. Modified lines are postfixed by a "prime", and new lines are numbered Nxy.

Each process p_i manages a new local variable ak_i (approximate k), initialized to $\sigma_i = \lceil \frac{\Delta_i}{m} \rceil + 1$, and whose final value will be K. The additional behavior of p_i is now as follows.

- First, when a non-root process informs its parent of its local termination, it now broadcasts the message TERM$(parent_i, id_i, ak_i)$ (line 28'). Hence, the values ak of the leaves will be the first to be known by their parent.
- When a process receives a message TERM$(dest, id, ak)$ (line 30'), in addition to its previous statements, it updates ak_i to $\mathsf{max}(ak_i, ak)$ (line N1). It follows that the root learns the value of K (which is the maximal value ak it receives). When it learns it, the root progresses to the local state 5 (line 34').
- Starting from the root, when a non-leaf process is in state 5 and allowed to broadcast (predicate of line N2), it broadcasts the message END(id_i, ak) and progresses to the final state 6 (line N3).
- Finally, when a process p_i, in state 4, receives a message END$(sender, ak)$ from its parent, it updates ak_i to $\mathsf{max}(ak_i, ak)$ and progresses to state 5 (line N6). It will then forward the message END$(sender, ak)$ to its children if it has some, and in all cases will enter local state 6 (line N3).

After the local state of a process p_i became 6, we have $ak_i = K$. Hence, all processes are provided with a round-based programming level in which each process p_i can C2m-freely broadcast messages at all the rounds such that $(CLOCK \bmod ak_i) \in colors_i$.

```
(28')       else broadcast TERM(parent_i, id_i, kprime); state_i ← 4

(30')  when TERM(dest, id, ak) is received do
(31)     if (dest ≠ id_i) then discard the message (do not execute lines 31-35) end if;
(32)     colored_i ← to_colored_i ∪ {id};
(N1)     ak_i ← max(ak_i, ak);
(33)     if (colored_i = neighbors_i)
(34')       then if (parent_i = id_i) then state_i ← 5 else state_i ← 3 end if
(35)     end if.

(N2)  when ((CLOCK mod ak_i) ∈ color_i) ∧ (state_i = 5)) do
(N3)     if (|neighbors_i| ≠ 1) then broadcast END(id_i, k_i) end if; state_i ← 6.

(N4)  when END(sender, k) is received do
(N5)     if (sender = parent_i) ∧ (state_i = 4)
(N6)       then k_i ← max(k_i, k); state_i ← 5
(N7)     end if.
```

Algorithm 3: Obtaining the value $K = \lceil \frac{\Delta}{m} \rceil + 1$ and informing all processes